A Plumbers Approach to Symplectic Geometry, Part I

1/21/15

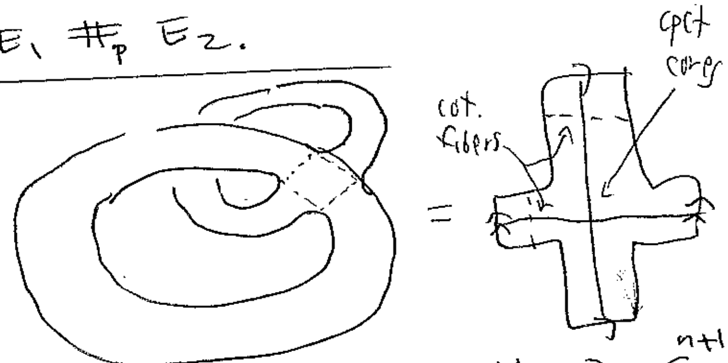
$\mathbb{D}^n \to E_1^{2n} \downarrow Q_1^n$ $\mathbb{D}^n \to E_2^{2n} \downarrow Q_2^n$

Plumbing: Consider disk bundles $\mathbb{D}^n / Q_1^n$ and $\mathbb{D}^n / Q_2^n$

Each has a neighborhood diffeo to $\mathbb{D}^n \times \mathbb{D}^n$
Pick a diffeo $\mathbb{D}^n \times \mathbb{D}^n \to \mathbb{D}^n \times \mathbb{D}^n$ which exchanges the two factors, and use this to ~~identify~~ glue E_1^{2n} to E_2^{2n} $\rightsquigarrow$ plumbing of E_1 and E_2, $E_1 \#_p E_2$.

Ex: $T^* S^1 \#_p T^* S^1$:

(A_2)



Ex: Milnor fiber $A_m^{2n} = \{ z_1^2 + z_2^2 + \dots + z_n^2 + t^{m+1} = 1 \} \subset \mathbb{C}^{n+1}$ is a plumbing of T^*S^n's according to the graph

Ex: A_3:

Fact: $T^* Q_1^n \#_p T^* Q_2^n$ has a natural L'ville (and in particular symplectic) str

~~Main results (Abouzaid–Smith)~~ Wrapping

~~Recall~~: • Let (M, λ) be a L'ville manifold, i.e. $d\lambda$ is symplectic and the end is of the form $(\mathbb{R}^+_r \times \partial_e M, e^r \lambda|_{\partial M})$.

• In particular, $(\partial M, \lambda)$ is a ctct str.

• For $L, L' \subset M$ Lagns (w/ Legn bdy if noncpct), define
$$CW(L, L') = \mathbb{Z}\langle \Phi(L) \cap L' \rangle,$$
where Φ is the Hamilton flow of a function H which looks like $(e^r)^2$ on the collar.

• Equip CW with the usual Floer diff'tl counting holom. disks

$\rightsquigarrow$ "wrapped Floer homology" & "wrapped Fukaya category"

(note: will assume all A_∞ cats, modules, etc are minimal)

Rmk: For L, L' cpct, agrees w/ usual Floer homology.

Ex: For $T^*_q Q \subset T^*Q$ a cot. fiber, $\binom{Q}{\text{spin}}$
$$HW(T^*_q Q, T^*_q Q) \cong H_{-*}(\underbrace{\Omega_q Q}_{\text{based loop space.}})$$

In fact,
$$\underbrace{CW(T^*_q Q, T^*_q Q)}_{A_\infty\text{-algebra}} \underset{\substack{A_\infty \\ \text{equivalence}}}{\cong} C_{-*}(\Omega_q Q)$$

Local systems:

Def: A local system on a Lagn L is a bundle of Abelian gps over L, i.e. a $\mathbb{Z}[\pi_1(L)]$-module.

For $\begin{matrix} E \\ \downarrow \\ L \end{matrix}$, $\begin{matrix} E' \\ \downarrow \\ L' \end{matrix}$ Lagns w/ local systems,

can define $HF(E, E')$, using $\bigoplus_{p \in L \cap L'} \hom(E_p, E'_p)$ and using holonomy along sides of a strip  to define the diff'le

Twisted complexes:

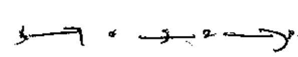
$$Fuk(M) \rightsquigarrow \underbrace{Tw\,Fuk(M)}_{\text{enlarged } A_\infty \text{ category}}$$

Elts of $Tw\,Fuk(M)$ consist of

- a formal direct sum $V_1 \otimes L_1 \oplus \dots \oplus V_k \otimes L_k$
- V_i graded Ab. gp
- L_i Lagn
- an internal diff'le δ_{ij}, where for $i < j$, $\delta_{ij} \in \hom(V_i, V_j) \otimes CF(L_i, L_j)$
- δ_{ij} satisfies a "generalized MC eqn"

$$\sum_{r=1}^{\infty} \mu^r(\delta, \dots, \delta) = 0$$

Rmk: Arrows look like  instead of just



• Morphisms in TwFuk: made of morphisms b/w ~~sums~~ summands

Ex: $\mathrm{Mor}_{\mathrm{TwFuk}}(V_1 \otimes L_1 \oplus V_2 \otimes L_2,\ W \otimes Q)$

$$= \hom(V_1, W) \otimes CF(L_1, Q)$$
$$\oplus \hom(V_2, W) \otimes CF(L_2, Q)$$

• A_∞ op'ns in Tw Fuk come from those of Fuk, twisted by internal diff'tls:

$$\mu^r_{\mathrm{TwFuk}}(\cancel{\delta,} f_d, \dots, f_1)$$

$$:= \sum_{i_0, \dots, i_d} \mu^{d+i_0+\dots+i_d}(\underbrace{\delta, \dots, \delta}_{i_d}, f_d, \delta \dots, \delta, f_{d-1}, \dots, f_1, \underbrace{\delta \dots, \delta}_{i_0})$$

Main results: (Abouzaid - ~~Seidel~~ Smith)

Let $M = T^*Q_1^n \#_p T^*Q_2^n$

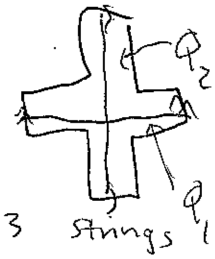
(results will also hold for plumbing along a tree)

Let $\mathcal{W}(M)$ = wrapped Fuk. cat
$\mathcal{F}(M)$ = Fuk. cat. of closed, exact Lag'ns

Thm: $\mathcal{F}(M) \subset \mathcal{W}(M)$ is gen'd by the cot. fibers $T^*_{q_1} Q_1$ and $T^*_{q_2} Q_2$.

Thm & If $\dim_{\mathbb{R}} M \geq 6$, every closed exact Maslov zero Lagn is equivalent to a tw. cpx over the cpct cores equipped w/ local systems.

Now consider $A_2^{2n \geq 6}$



Fact: $\underbrace{\langle \tau_{Q_1}, \tau_{Q_2} \rangle}_{\text{Dehn twists}} \cong \underbrace{Br_3}_{\text{braid gp of 3 strings}}$

Thm: Every ~~exact~~ ~~closed exact~~ Maslov zero Lagn in $\mathcal{F}(A_2^{2n})$ lies in the orbit of one of the cpct cores under Br_3-action.

Cor: Every closed exact Maslov zero Lagn in A_2^n is a homology sphere and lies in a primitive homology class.

The fibers and the diagonal generate $\mathcal{F}(\underbrace{M}_{T^*Q_1 \#_p T^*Q_2})$

Let $Q = Q_1 \# Q_2$, and $\underset{S^{n-1}}{V} \subset Q$ the belt sphere of the connect-sum.

Pick a Morse function $\phi : Q \to \mathbb{R}$ s.t.

- the Q_1^{int} region maps to negative reals
- Q_2^{int} region maps to positive reals
- $V = \phi^{-1}(0)$ is a regular level set

$\left(\partial Q_1^{int} = \partial Q_2^{int} = V \right)$

Fukaya–Seidel–Smith: $\phi: Q^n \to \mathbb{R}$ extends to a Lefshetz fibration

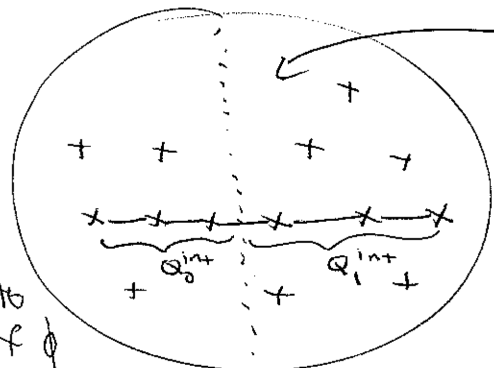
$$\begin{array}{ccc} Q_{\mathbb{C}}^{2n} & \overset{\text{exact Lagn}}{\supset} & Q \\ \tilde{\phi}_{\mathbb{C}} \downarrow & & \\ \mathbb{C} & & \end{array}$$

This means:

- $Q_{\mathbb{C}}$ is Liville, as are the fibers of $\tilde{\phi}_{\mathbb{C}}$
- $\tilde{\phi}_{\mathbb{C}}$ is a submersion onto its image, except for isolated c.p.'s modeled on $\mathbb{C}^n \to \mathbb{C}$, $(z_1, \dots, z_n) \mapsto z_1^2 + \dots + z_n^2$

Image of $\phi_{\mathbb{C}}$:

real c.v.'s corresp. to those of ϕ



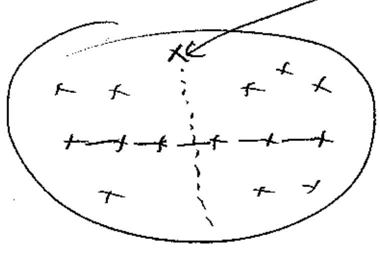
can assume no c.v.'s on y-axis

Note: $V \subset F = \phi_{\mathbb{C}}^{-1}(0)$ is a Lagn $(n-1)$-sphere

We now add a vanishing cycle to $\phi_{\mathbb{C}}$ which corresponds to V to get a new L.f. $\Phi: E \to \mathbb{C}$

Image of Φ:

new v.c. at $(0,1)$

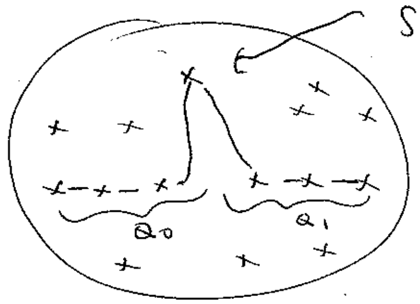


Let T be the set of pts given by parallel transporting V along the segment $\{(0,y) : y \in [0,1]\}$.

Then T is a Lagn disk w/ $T \cap F = V$.

T is called the thimble of the v.c. at $(0,1)$

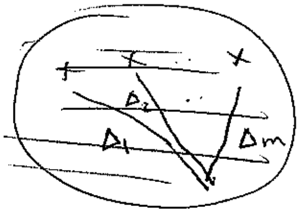
Inside E, have a model for the plumbing:

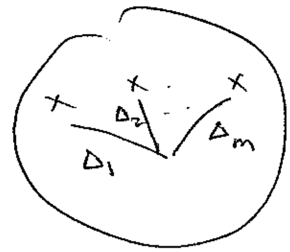


Should both be horizontal lines near $(0,1)$

Generating $\mathcal{F}$(Lefshetz fibration) (closed, exact Lagns)

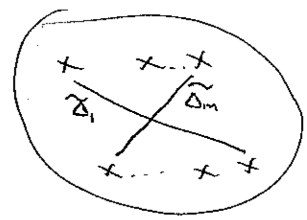
Consider a general Lef. fibr. $X \to \mathbb{D}^2$ w/ basis of thimbles $\Delta_1, \dots, \Delta_m$.





Seidel's trick: Pull back $X \to \mathbb{D}^2$ along a branched double cover $\mathbb{D}^2 \to \mathbb{D}^2$, ~~(i.e. $z \mapsto z^2$)~~ $z \mapsto z^2$.

Get new L.F. $\widetilde{X} \to \mathbb{D}^2$ w/ twice as many V.C.s



s.t.

- $X \subset \widetilde{X}$ L'ville subdomain
- $\widetilde{X}$ contains Lagn spheres (matching cycles) $\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_m$, w/ $\widetilde{\Delta}_i \cap X = \Delta_i$
- $\tau_{\widetilde{\Delta}_1} \circ \tau_{\widetilde{\Delta}_2} \circ \dots \circ \tau_{\widetilde{\Delta}_m}$ = covering transf. of $\widetilde{X} \to X$

  In particular, sends E to a region disj. from X.

Let $L \subset \cancel{X}$ be a closed Lagn.

Recall: $\tau_{\tilde{\Delta}_i} L \cong \underbrace{HF(\tilde{\Delta}_i, L) \otimes \tilde{\Delta}_i \xrightarrow{ev} L}_{\text{a twisted cpx}}$

Have $\tau_{\tilde{\Delta}_1} \circ \cdots \circ \tau_{\tilde{\Delta}_m} L = \underset{\sim}{L_-}$

a Lagn in $\tilde{X}$ disj. from $\cancel{X}$

Hence $L_- \cong \mathrm{Cone}(C \to L)$,

where C is in the subcat of $\mathcal{F}(\tilde{X})$ gen'd by $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_m$.

Open string Viterbo:

Let (W, λ) be L'ville, and assume $W^{in} \subset W$ is a codim 0 submfd, and (W^{in}, λ) is also L'ville.

Prop (Abouzaid–Seidel): There is a A_∞ functor $\mathcal{W}(W) \to \mathcal{W}(W^{in})$ which assigns to a Lagn its intersection w/ W^{in}

Rmk: ~~Need ... to assume elts of~~ $\cancel{\mathcal{W}(W^{in})}$

$\boxed{\text{Technical hyp:}}$ Should restrict to elts of $\mathcal{W}(W^{in}), \mathcal{W}(W)$ having a locally const. primitive near the bdy, and elts of $\mathcal{W}(W)$ should also have a primitive which is constant near ∂W^{in}.

Rmk. Compare to Viterbo restr. for SH.

Above, had $\forall$ ~~\$~~ closed Lagn in $\tilde{E}$,

$$\operatorname{Cone}(\underbrace{C}_{\substack{\text{gen'd} \\ \text{by } \tilde{\Delta}_1,\dots,\tilde{\Delta}_m}} \to L) \cong \underbrace{L_-}_{\substack{\text{Lagn in } \tilde{E} \text{ disj.} \\ \text{from } \tilde{E}}}$$

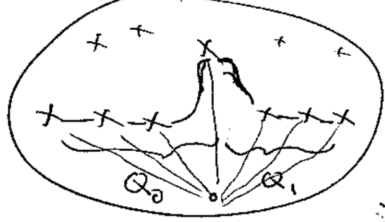
Apply Viterbo:

- L_- goes to 0
- C goes to smth gen'd by $\Delta_1,\dots,\Delta_m$
- L goes to itself

Conclusion: ~~The Fukaya cat. of closed Lagns in E~~

$\underbrace{\mathcal{F}(\tilde{E})}_{\substack{\text{closed} \\ \text{Lagns}}} \subset \mathcal{W}(\tilde{E})$ is gen'd by $\Delta_1,\dots,\Delta_m$

Back to our specific Lf: $\Phi: E \to \mathbb{C}$



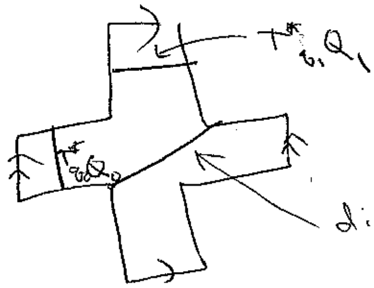
Consider the thimbles.

~~$\exists$~~ $\exists$ a Weinst. nbd of $Q_0 \cup Q_1$ s.t. ~~they intersect it in a cot fiber of Q_0 or Q_1 or~~

- the c.p.s on Q_0^{int} or Q_1^{int} have thimbles given by cot. fibers of T^*Q_0, T^*Q_1 resp.
- Near the ~~c.p.~~ c.v. at $(0,1)$, $Q_0 \cup Q_1$ looks like $\mathbb{R}^n \cup i\mathbb{R}^n \subset \mathbb{C}^n$, and the thimble looks like the diagonal $(1+i)\mathbb{R}^n$

Apply Viterbo to conclude:

<u>Prop</u> $\mathcal{F}(T^*Q_0 \#_p T^*Q_1)$ is gen'd by $T^*_{q_0}Q_0$, $T^*_{q_1}Q_1$, and the diagonal



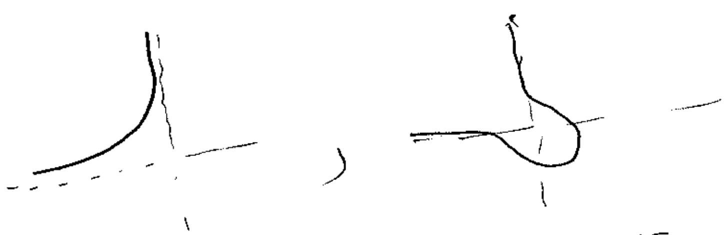
<u>Generating the diagonal</u>

<u>Lagrange surgery</u>:

curve in $\mathbb{C}$ $\rightsquigarrow$ $V_\gamma = \bigcup_{t \in \mathrm{im}(\gamma)} t \cdot S^{n-1} \subset \mathbb{C}^n$

Lag'n submfd.

$\leftrightsquigarrow$ $\mathbb{R}^n \cup i\mathbb{R}^n \subset \mathbb{C}^n$



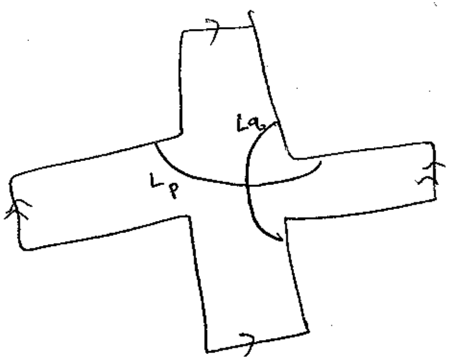
$\Leftarrow$ $\Big\}$ agree w/ $\mathbb{R}^n \cup i\mathbb{R}^n$ outside a cpct set

"positive Lag'n surgery"

"negative Lag'n surgery"

$\rightsquigarrow$ can resolve two Lag'ns intersecting transv. at one pt to get a smooth Lag'n

Consider L_p, L_q intersecting cot. fibers



(i.e. just $T^*_{q_0} Q_0$ and $T^*_{q_1} Q_1$)

<u>Fact</u>: $\mathrm{Cone}(L_p \to L_q)$ and $\mathrm{Cone}(L_q \to L_p)$ are quasi-iso to the two Lagrange surgeries

<u>pf</u>: Seidel's ~~exact triangle~~ charac. of Dehn twists, together w/ relation b/w D.t.'s and Lag'n surgery, and apply Viterbo

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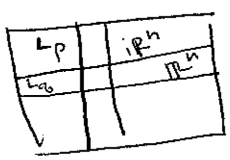
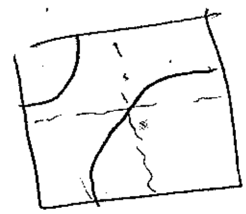
In model plumbing region:



$\rightsquigarrow$ Lag'n surgery



<u>Observe</u>: Lag. surgery of L_q w/ L_p agrees w/ diagonal near $\mathbb{R}^n \cup i\mathbb{R}^n$.

Then applying Viterbo to smaller nbd of $Q_0 \cup Q_1$ in E,

$\mathrm{Cone}(L_p \to L_q) \simeq L.S.(L_q, L_p)$

$\Rightarrow$ diagonal gen'd by $T^*_{q_0} Q_0$ and $T^*_{q_1} Q_1$

Generation by compact cores:

Prop: $\overset{\dim 2n}{A}$ closed ~~exact~~ nbhd of Lagn $L \subset T^*_{Q_0} \#_p T^*_{Q_1}$ is gen'd by finite rank local systems over Q_0 and Q_1.

$\underbrace{\phantom{T^*_{Q_0}\#_p T^*_{Q_1}}}_{M}$

Let $A^* = \bigoplus_{i,j} CW^*(L_{q_i}, L_{q_j})$.

Generation of $\mathcal{W}(M)$ by cot. fibers

$\Rightarrow \mathcal{W}(M) \hookrightarrow$ cat. of A_∞-modules over A^*

So suffice to understand A^*-modules supp. in finitely many cohom. degrees.

Recall:

$HW^*\left(\begin{matrix}\text{cot. fiber in} \\ \text{cotangent bundle}\end{matrix}\right) \cong H_{-*}\left(\begin{matrix}\text{based} \\ \text{loops}\end{matrix}\right)$

supp. in non-pos degrees.

Similarly, have computation: (note: dont actually need to compute all of A^*!)

Prop: • A^* is supp. in non-pos. degrees for $n \geq 3$ (false for $n=1$!)

• $A^0 = \bigoplus_i CW^0(L_{q_i}, L_{q_i})$

Note: Local system $\begin{matrix} \nu \\ \downarrow \\ Q_i \end{matrix} \rightsquigarrow \overset{[\text{right}]}{A^*}\text{-module } (CW^*(L_{q_i}, \nu))$

(Assume $n \geq 3$ from now on)

Prop: $CW^0(L_{q_i}, L_{q_i}) \cong \mathbb{Z}[\pi_1(Q_i)]$

(as rings)

$CW^0(L_{q_i}, L_{q_i})$ - module $CW^*(L_{q_i}, ?)$ $\longleftrightarrow$ $\mathbb{Z}[\pi_1(Q_i)]$ - module ?

Now consider an A^* - module P^* supp. in f.m. cohom. degrees.

- Since A^* ~~is~~ non-pos graded,

P^* has filtration $P^{\leq k} := \bigcup_{* \leq k} P^*$

since • $\mu^{1|d} : P \otimes A^d \to P$ has degree $1-d$

and • can assume P minimal, i.e. $\mu^{1|0} \equiv 0$

- The quotients P^{k+1}/P^k are precisely A^0 - modules.

Since $A^0 \cong \bigoplus_i \mathbb{Z}\pi_1(Q_i)$, an A^0 - module exactly corresp. to a local system over Q_0 and a local sys. over Q_1.

$\square$

For part III, quick review:

~~Thm.~~ ~~[illegible]~~

- Let Q_0, Q_1 be n-mfds for $n \geq 3$.
- $\mathcal{F}$ = Fuk cat of closed, exact Lagns in $T^*Q_0 \#_p T^*Q_1$.

Thm: Every closed, exact, Maslov 0 Lagn in $T^*Q_0 \#_p T^*Q_1$ is $\cong$ ~~equivalent~~ ~~in $\mathcal{F}$ to a twisted cpx~~ generated by Q_0 and Q_1, equipped with finite rank local systems.

Pf: • Equip $M = T^*Q_0 \#_p T^*Q_1$ w/ nice Lef. fibr.

plus ~~[illegible]~~ open-string Viterbo functor

- General thry of Lef. fibrs: M gen'd by $T^*_{q_0}Q_0$, $T^*_{q_1}Q_1$ and the diagonal.
- Seidel's LES $\Rightarrow$ diagonal is redundant.
- $A^* := \bigoplus_{i,j} CW^*(T^*_{q_i}Q_i, T^*_{q_j}Q_j)$

is non-pos. graded and

$$A^0 \cong \mathbb{Z}[\pi_1(Q_0)] \otimes \mathbb{Z}[\pi_1(Q_1)]$$

(as rings) gives the result.

Rmk: A^* is not non-pos graded in the case $M = T^*S^1 \#_p T^*S^1 = T^2 \setminus \{pt\}$

From now on, dim ≥ 3

We've shown that any closed, exact, Maslov zero Lagn $L \subset T^*Q_0 \#_p T^*Q_1$ is generated by finite rk local systems on Q_0 and Q_1.

We now explain: Fix a coeff. field $\mathbb{K}$.

~~Thm~~: Let $\Sigma_0^{n\geq 3}, \Sigma_1^{n\geq 3}$ be $\mathbb{Z}H$-spheres and assume $\pi_1(\Sigma_i)$ has no non-trivial finite dim'l repns.

Have a Br_3 action on $Tw\,\mathcal{F}(T^*\Sigma_0 \#_p T^*\Sigma_1)$ by $\langle T_1, T_2\rangle$ (recall $\mathcal{F}$ = closed Lagns)

Here T_i is the twist functor

$$T_i: \quad L \mapsto \underbrace{CF(\Sigma_i, L)\otimes \Sigma_i \xrightarrow{ev} L}_{\text{tw. cpx}}$$

Rmk:
- Can define twist functor T_A $\forall$ object A
- If $HF^*(A,A) \cong H^*(S^n)$ (i.e. A is a $\mathbb{Z}H$-sphere) then T_A is invertible,

$$\text{w/ } T_A^{-1}: L \mapsto \underbrace{L \xrightarrow{ev^\vee} CF(L,A)^\vee \otimes A}_{\text{tw cpx}}$$

- If A is a Lagn sphere, $T_A \simeq \tau_A$

geometric Dehn twist

Thm: Modulo quasi-equivalence and grading shift in $Tw\,\mathcal{F}$, every closed Lagn L lies in the $Br_3 = \langle T_1, T_2\rangle$ orbit of Q_0.

Ex: A_2-Milnor fiber.

Picking gradings, can assume

$$HF(Q_i, Q_i) \cong \mathbb{K}\langle \underset{\substack{\uparrow \\ \deg 0 \\ \text{identity morphism}}}{e_i}, \underset{\substack{\uparrow \\ \deg n \\ \text{"fund. class"}}}{f_i} \rangle$$

$$HF(Q_0, Q_1) \cong \mathbb{K}\langle \underset{\substack{\uparrow \\ \deg 1}}{p} \rangle$$

$$\left. HF(Q_1, Q_0) \cong \mathbb{K}\langle \underset{\substack{\uparrow \\ \deg -n-1}}{p^{\vee}} \rangle \right\} \text{Poincaré duality for } HF$$

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* Assume $\mathcal{F}$ is minimal. Also, strictly unital. *

Using the fact that A^* is non-pos graded, can prove

Lemma: Any Lagn $L \in \mathcal{F}$ is equiv. to a tw. cpx built from Q_0, Q_1 (no l.c.s) such that none of the arrows are e_0 or e_1.

Observe: The higher products $\mu^{k \geq 3}$ b/w Q_0, Q_1 vanish for degree reasons. Hence only have μ^2.

Ex: $\mu^3(p, p^{\vee}, p) \in HF^{(1+n-1+1)-1}(Q_0, Q_1)$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$$

$$HF^n(Q_0, Q_1) = 0$$

Can now assume: L is a tw. cpx with degrees in $[0, N]$, and no e_i arrows

Ex: For $n=4$,

$$
\begin{tikzcd}
L \simeq & U_0\otimes Q_0 \arrow[d, "f_{00}\otimes p"] & U_1\otimes Q_0 \arrow[d, "f_{11}\otimes p"] & U_2\otimes Q_0 \arrow[d, "f_{22}\otimes p"] & U_3\otimes Q_0 \arrow[d, "f_{33}\otimes p"] \arrow[lll, bend right, "h_{30}\otimes f_0"'] \\
& V_0\otimes Q_1 & V_1\otimes Q_1 \arrow[ul, dashed] & V_2\otimes Q_1 \arrow[ul, dashed, "g_{20}\otimes p^\vee"] & V_3\otimes Q_1 \arrow[ul, dashed, "g_{31}\otimes p^\vee"] \arrow[lll, bend left, "K_{30}\otimes f_1"]
\end{tikzcd}
$$

- Arrows are v.s. homs

i.e. $\downarrow$ labeled by p
$\dashleftarrow$ " " $p^\vee$
$\curvearrowleft$ " " f_0
$\curvearrowleft$ " " f_1

Here f_{ij}, g_{ij}, h_{ij} are v.s. homomorphisms

Can check: All arrows have degree 1.

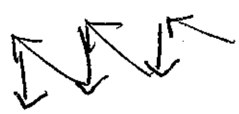
Idea: Apply T_0, T_1, T_0^{-1}, or T_1^{-1} to simplify the tw. cpx for L.

That is, we'll apply ~~total~~ induction on the complexity:

Def: Let $|U_i| = 2i+1$, $|V_j| = 2j$,

$$Cx(L) := \max\left(\max_{U_i\neq 0}|U_i|, \max_{V_i\neq 0}|V_i|\right) - \min\left(\min_{U_i\neq 0}|U_i|, \min_{V_i\neq 0}|V_i|\right)$$

Rmk: ~~In~~ Cx is the longest zig-zag



with nonzero starting and ending pts.

Ex $Cx\begin{pmatrix} U_0 & U_1 \\ V_0 & V_1 \end{pmatrix} = \#(\downarrow\nwarrow\downarrow) = 3$

Ex $Cx(L) = 7$

** ~~[illegible]~~ <u>Important obs</u>: $HF^*(L,L)$ is non-neg. graded.

$$\begin{array}{c} HF^*(L,L) \\ \cong \\ H^*(L) \end{array}$$

Therefore an elt of $CF^*(L,L)$ which is closed, non-exact, and has neg degree $\rightsquigarrow$ contradiction

<u>Claim</u>: Assume $V_0 \neq 0$.

Then

$$\begin{tikzcd} U_0 \arrow[d] & U_1 \arrow[d] \arrow[l, dashed] & U_2 \arrow[d] \arrow[l, dashed] & U_3 \arrow[d] \arrow[lll, bend right] \\ V_0 & V_1 \arrow[l, dashed] & V_2 \arrow[l, dashed] & V_3 \arrow[l, dashed] \arrow[lll, bend left] \end{tikzcd}$$

} Surjective v.s. homs

<u>pf</u>: Suppose say $U_3 \to V_3$ not surj. (by contradiction).

Then $V_3 \cong \mathrm{im}(U_3) \oplus V_3'$.

Consider a nonzero elt q of $\mathrm{Hom}(V_3', V_0) \otimes e_1$

i.e. $q \in CF^{-3}(L,L)$.

internal diff on ch. cpx for L

Then $\mu^1_{Tw}(q) = \mu^2(\delta, q) + \mu^2(q, \delta)$

by defn of μ^1_{Tw} $= 0$ since no arrows hit V_3' or emanate from V_0.

Also, q <u>not</u> exact, since

$\mu^2(a,b) = e_1 \Rightarrow a = b = e_1$, for degree reasons

but no e_i's in the tw. cpx for L!

But q has degree $-3 < 0 \Rightarrow$ contradiction.

Can similarly show that

- $U_i \hookrightarrow V_i$ for $i = 0, 1, 2$

- $U_0 \leftarrow V_2$ and $U_1 \leftarrow V_3$ are zero.

So have

$$L \simeq \begin{tikzcd} U_0 \arrow[d, hook] & U_1 \arrow[d, "\cong"] & U_2 \arrow[d, "\cong"] & U_3 \arrow[d] \arrow[lll, bend right] \\ V_0 & V_1 & V_2 & V_3 \arrow[lll, bend left] \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} U_0 \arrow[d, hook] & U_1 \arrow[d, equal] & U_2 \arrow[d, equal] & U_3' \oplus V_3 \arrow[d] \arrow[lll, bend right] \\ U_0 \oplus V_0' & U_1 & U_2 & V_3 \arrow[lll, bend left] \end{tikzcd}$$

Recall: $T_i(Q_i) \simeq Q_i[1-n]$

$\Rightarrow T_i^{-1}(Q_i) \simeq Q_i[n-1]$

$\Bigg\}$ maybe slightly surprising at first glance

Now apply T_0^{-1} to L:

- $T_0^{-1}(U_3' \otimes Q_0) = U_3' \otimes Q_0[3]$,

  i.e. U_3' moves to the left 3 positions

- $T_0^{-1}\left(V_3 \otimes (Q_0 \xrightarrow{P} Q_1)\right)$

$$\simeq T_0^{-1}(V_3 \otimes T_0 Q_1)$$

$$\simeq V_3 \otimes Q_1,$$

i.e.
$$\begin{array}{ccc} V_3 & & 0 \\ \downarrow & \text{becomes} & \\ V_3 & & V_3 \end{array}$$

- Similarly,
$$\begin{array}{ccc} U_1 & & 0 \\ \| & \rightsquigarrow & \\ U_1 & & U_1 \end{array}$$

and
$$\begin{array}{ccc} U_2 & & 0 \\ \| & \rightsquigarrow & \\ U_2 & & U_2 \end{array}$$

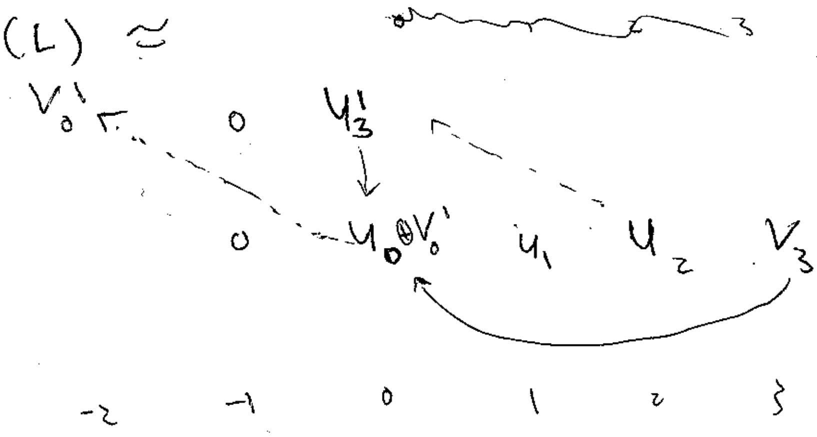
and $\overset{n-2}{+2}$??

Have $T_0^{-1}(Q_1) \simeq Q_1 \to Q_0[\cancel{2-n}]$, 2

so
$$\begin{array}{ccc} U_0 \otimes Q_0 & & \\ \downarrow & \rightsquigarrow & \\ (U_0 \oplus V_0') \otimes Q_1 & & U_0 \otimes Q_1 \oplus V_0' \otimes Q_1 \end{array}$$

$V_0' \otimes Q_0[\cancel{2}]$ 2

So $T_0^{-1}(L) \simeq$



$cx(T_0^{-1}(L))$ is not $<$ $cx(L)$,
but it is if $V_2 = V_3 = 0$.

~~For eg, not~~

Note, for eg, then $V_3 \neq 0 \Rightarrow V_0 = 0$,
since can take a nonzero elt $\underbrace{\hom(V_3, V_0) \otimes e_1}_{\text{closed, non-exact, neg deg}}$

not supposed to be ~~false~~ obvious!

Claim: ~~If~~ V_2, V_3 not both zero,
$cx(T_1(L)) < cx(L)$

Finally, if $V_0 = 0$, there's a similar analysis in which we apply T_1^{-1} or T_0 instead.

Eventually, get $cx = 1 \Rightarrow V_0 \otimes Q_0$ or $V_1 \otimes Q_1$
But $HF^0(L, L)$ is 1-dim
$\Rightarrow V$ or V is 1-dim'l.
Also, note that Q_0, Q_1 lie in same Br_3 orbit!
(since $T_{Q_0} Q_1 \simeq t_{a_1}^{1} Q_0$, at least up to shift) $\boxed{\equiv}$

Spherical twists

Recall: T^*S^n admits a geometric Dehn twist b/c S^n admits periodic geodesic flow.

Also have such twists for $\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n, \mathbb{OP}^2$.

Thm (Bott) M admits periodic geod. flow $\Rightarrow H^*(M)$ is a truncated poly-ring

For Σ not diffeo to S^n, T_Σ has no geometric candidate.

Expect: For Σ $\mathbb{Z}H$-sphere:

$$\pi_0 \mathrm{Symp}_{ct}(T^*\Sigma) \overset{\text{image}}{\subset} \mathrm{Aut}(\mathcal{F}(T^*\Sigma))/\langle[1]\rangle$$

$$\supset \langle T_\Sigma \rangle$$

intersect only in identity.

Problem: $\mathcal{F}(T^*\Sigma)$ has only one object, so need to enlarge somehow.
(one approach: Nadler-Zaslow)

Let $M_\Sigma = T^*\Sigma \#_p T^*S^n = T^*\Sigma \cup$ crit. handle

Lemma: $T_\Sigma \in \mathrm{Aut}\,\mathcal{F}(M_\Sigma)/\langle[1]\rangle$ has infinite order.

Pf: $\mathrm{rk}\, HF(S^n, T_\Sigma^k(S^n)) \to \infty$ as $k \to \infty$.

Thm: For $\pi_1(\Sigma) \neq 1$,

$$\pi_0 \, \mathrm{Symp}_{ct}(T^*\Sigma) \overset{\text{image}}{\subset} \mathrm{Aut}\, \mathcal{F}(M_\Sigma) / \langle [1] \rangle$$
$$\cup$$
$$\langle T_\Sigma \rangle$$

meet only in the identity.
(i.e. T_Σ is not geometric!)

will use:

Thm (Abouzaid) Let M be Liouville and $\pi : \widetilde{M} \to M$ the univ. cover.
There is an A_∞ category $\mathcal{W}(\widetilde{M}; \pi)$ and a pullback functor

$$\pi^* : \mathcal{W}(M) \to \mathcal{W}(\widetilde{M}; \pi)$$

- sending L to $\pi^{-1}(L)$
- and s.t.

$$\begin{array}{ccc} HF^*(L,L) & \to & HF^*(\pi^{-1}(L), \pi^{-1}(L)) \\ \cong & & \cong \\ H^*(L) & \xrightarrow{\pi^*} & H^*(\pi^{-1}(L)) \end{array}$$ for L closed

- $\pi_1(M)$ acts by autos of $\mathcal{W}(\widetilde{M}; \pi)$

Rmk: $\mathcal{W}(\widetilde{M}; \pi)$ agrees w/ $\mathcal{W}(\widetilde{M})$ when $\pi_1(M) < \infty$.

pf of thm about T_Σ:

$$\text{Let } \begin{array}{c}\tilde{\Sigma}\\ \downarrow \text{univ. cover}\\ \Sigma\end{array} \rightsquigarrow \begin{array}{c}\tilde{M}_\Sigma\\ \downarrow \text{univ. cover}\\ M_\Sigma\end{array}$$

Suppose by contr. that T_Σ is geometric.

Then

$$L := "\tau_\Sigma^{-2}(S^n)" \simeq \cancel{\Sigma} \; T_\Sigma^{-2}(S^n)$$

$$\simeq \Sigma[n-1] \xleftarrow{[\Sigma]} \Sigma \leftarrow S^n$$

geometric Lag'n

* Pick coeff field $\mathbb{K}$ s.t. Char($\mathbb{K}$) divides $|\pi_1(\Sigma)|$ (vacuous if $|\pi_1(\Sigma)| = \infty$) *

Applying π^{-1}, get

$$\pi^{-1}(L) \simeq \tilde{\Sigma}[n-1] \xleftarrow{0} \tilde{\Sigma} \leftarrow \pi^{-1}(S^n)$$

$$\simeq \tilde{\Sigma}[n-1] \oplus (\tilde{\Sigma} \leftarrow \pi^{-1}(S^n))$$

Note: $\tilde{\Sigma}$ connected $\Rightarrow$ indecomposable.

<u>Claim</u>: $\tilde{\Sigma} \leftarrow \pi^{-1}(S^n)$ also indecomp.

<u>Lemma</u>: In $\mathcal{W}(\tilde{M}_\Sigma; \pi)$, $\tilde{\Sigma}[n-1]$ and $\tilde{\Sigma} \leftarrow \pi^{-1}(S^n)$ are not in same deck transf orbit.

<u>pf</u>: Applying $HF(-, \text{a component of } \pi^{-1}(S^n))$, get different ranks.

But the components ~~of~~ $\widetilde{L}_\alpha$ of $\pi^{-1}(L)$ are
- all related by deck transf.
- each indecomp.

~~@~~ Claim: over $\mathbb{K}$, the indecomp. decomposition is unique.

This contradicts

$$\pi^{-1}(L) \cong \widetilde{\Sigma} \oplus \underbrace{(\widetilde{\Sigma} \leftarrow \pi^{-1}(S^n))}$$

$\sim$

indecomp, not related by deck transf

$\boxed{\smile}$

With scalar techniques, can prove:

<u>Thm</u>: Let Q be a simply-conn 4-mfd. Suppose T^*Q has a cptly supp. symplecto acting non-trivially on objects of $\mathcal{L}(M_Q)$.

Then $Q \underset{\text{htpy eq.}}{\simeq} S^4$ or $\mathbb{CP}^2$.